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Bifurcation Stabilization with Local Output Feedback *

Guoxiang Gu†, Xiang Chen†, Andrew G. Sparks‡, and Siva S. Banda‡

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Abstract

Local output feedback stabilization with smooth nonlinear controllers is studied for parameterized nonlinear systems for which the linearized system possesses either a simple zero eigenvalue, or a pair of imaginary eigenvalues, and the bifurcated solution is unstable at the critical value of the parameter. It is assumed that the unstable mode corresponding to the critical eigenvalue of the linearized system is not linearly controllable. Two results are established for bifurcation stabilization. The first one is stabilizability conditions for the case where the critical mode is not linearly observable through output measurement. It is shown that nonlinear controllers do not offer any advantage over the linear ones for bifurcation stabilization. The second one is stabilizability conditions for the case when the critical mode is linearly observable through output measurement. It is shown that linear controllers are adequate for stabilization of transcritical bifurcation, and quadratic controllers are adequate for stabilization of pitchfork and Hopf bifurcations, respectively. The results in this paper can be used to synthesize stabilizing controllers, if they exist.

1 Introduction

Stabilization of nonlinear control systems with smooth state feedback control has been studied by a number of people [3, 1, 2, 4, 9]. An interesting situation for nonlinear stabilization is when the linearized system has uncontrollable modes on imaginary axis with the rest of modes stable. This is so called *critical cases* for which the linear theory is inadequate. It becomes more intricate if the underlying nonlinear system involves a real-valued parameter. At critical values of the parameter, linearized system has unstable modes corresponding eigenvalues on imaginary axis, and additional equilibrium solutions will be born. The bifurcated solutions may, or may not be stable. The instability of the bifurcated solution may cause "hysteresis loop" in bifurcation diagram for both subcritical pitchfork bifurcation and Hopf bifurcation [6], and induce undesirable physical phenomina. Hence bifurcation stabilization is an important topic in nonlinear control.

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† Department of Electrical and Computer Engineering, Louisiana State University, Baton Rouge, LA 70803-5901.

‡ Control Analysis Section, WL/FIGC, 2210 Eighth Street, SUite 21, WPAFB, OH 45433-7531

Abed and Fu studied bifurcation stabilization using smooth local state feedback control [1, 2]. For Hopf bifurcation, stabilization conditions were obtained for both the case where the critical modes of the linearized system are controllable and uncontrollable. For stationary bifurcation, stabilization conditions were derived for the case where the critical mode of the linearized system is controllable. The uncontrollable case was investigated in [8] where normal forms of the nonlinear system are used. In this paper we study bifurcation stabilization via local output feedback controllers that are smooth. Throughout the paper it is assumed that the critical mode of the linearized system is uncontrollable. Moreover it is assumed that only output measurement, a linear combination of state variables, is available for feedback. It should be clear that in practice measurement of all state variables is unrealistic, especially when the underlying nonlinear system has high order. Often some of state variables are more expensive, or more difficult to measure than others. Hence the bifurcation stabilization problem studied in this paper has more engineering significance.

Two stabilization issues will be investigated. The first one regards bifurcation stabilization where the critical mode of the linearized system is unobservable through output measurement. Stabilizability conditions are established for both stationary bifurcation and Hopf bifurcation. In particular, it is shown that nonlinear controllers do not offer any advantage over the linear ones for bifurcation stabilization. Stabilizing controllers, if exist, can be taken as linear ones. The second one regards bifurcation stabilization where the critical mode of the linearized system is observable through output measurement. Observability is required if the previous stabilizability conditions fail that implies that additional sensors have to be deployed. Stabilizability conditions are also obtained in this case. It is shown that linear controllers are adequate for stabilization of transcritical bifurcation, and quadratic controllers are adequate for stabilization of pitchfork and Hopf bifurcations, respectively. The results also provide synthesis procedures for design of stabilizing controllers, if they exist.

The notations in this paper are standard. The collections of real and complex numbers are denoted by $\mathbf{R}$ and $\mathbf{C}$, respectively. If $c \in \mathbf{C}$, its complex conjugate is denoted by $\bar{c}$. The collection of real and complex vectors with size n are denoted by $\mathbf{R}^n$ and $\mathbf{C}^n$, respectively. A matrix M of size $p \times m$ can be viewed as a linear map from $\mathbf{C}^m$ to $\mathbf{C}^p$, and its transpose is denoted by M^T. For $m = p = n$, M is said to be stable, if all its eigenvalues are in the open left half plane.

2 Bifurcation Stability and Projection Method

This section considers the stability issue for bifurcated system using the projection method developed in [6]. The system under consideration is the following nth order parametrized nonlinear system:

$$\dot{x} = f(\gamma, x), \quad f(\gamma, 0) = 0 \;\; \forall \; \gamma, \tag{1}$$

where $x \in \mathbf{R}^n$ and γ is a real-valued parameter. It is assumed that $f(\cdot, \cdot)$ is sufficiently smooth such that the equilibrium solution x_e, satisfying $f(\gamma, x_e) = 0$, is a smooth function of γ. Since $f(\gamma, 0) = 0$ for all γ, $x_e = 0$ is a fixed equilibrium point which is called zero solution. The linearized system at the

zero solution is given by

$$\dot{x}_0 = A(\gamma)x_0, \quad A(\gamma) = \left.\frac{df(\gamma, x_e)}{dx_e}\right|_{x_e=0}. \tag{2}$$

Two different types of bifurcation will be considered in this paper and the determination of their local stability will be discussed in the next two subsections.

2.1 Local Stability for Stationary Bifurcation

For stationary bifurcation, it is assumed that $A(\gamma)$ possesses a simple eigenvalue $\lambda(\gamma)$, depend smoothly on γ, satisfying

$$\lambda(0) = 0, \quad \lambda'(0) = \frac{d\lambda}{d\gamma}(0) > 0, \tag{3}$$

while all other eigenvalues are stable in a neighborhood of $\gamma = 0$. It implies that the zero solution $x_e(\gamma) = 0$ is locally stable for $\gamma < 0$ and becomes unstable for $\gamma > 0$. Furthermore additional equilibrium solutions $x_e \neq 0$ will be born at $\gamma = 0$ that is a smooth function of γ by smoothness of $f(\cdot, \cdot)$. Such bifurcated solutions are independent of time t, and is called stationary bifurcation. Thus $\gamma = 0$ is the critical value of the parameter and $\lambda(\gamma)$ is called critical eigenvalue. The nonlinear system (1) at $\gamma = 0$ is referred to as critical system. The bifurcated solution of the nonlinear system born at $\gamma = 0$ may, or may not be locally stable. For simplicity, only double points [6] will be considered in this paper. It should be clear that stability of the bifurcated solution plays an important role for the dynamics of the nonlinear system, and is hinged to stability of the critical system. A useful tool to determine local stability of the bifurcated solution and of the critical system is the projection method developed in [6] and advocated in [2].

Let ℓ and r denote the left row and right column eigenvectors of $A(0)$, corresponding to the critical eigenvalue $\lambda(0) = 0$. Then $\ell r = 1$ by suitable normalization. Denote $\varepsilon = \ell x_e$, where $x_e \neq 0$ satisfying $f(\gamma, x_e) = 0$ is also an equilibrium solution of (1), or bifurcated solution. Then by [6] there exists a series expansion

$$\begin{bmatrix} x_e(\varepsilon) \\ \gamma(\varepsilon) \end{bmatrix} = \sum_{k=1}^{\infty} \begin{bmatrix} x_{ek} \\ \gamma_k \end{bmatrix} \varepsilon^k.$$

Since $f(\gamma, x)$ is sufficiently smooth, there exists Taylor expansion near the origin of $\mathbf{R}^n$ of the form

$$\dot{x} = f(\gamma, x) = L(\gamma)x + Q(\gamma)[x, x] + C(\gamma)[x, x, x] + \cdots \tag{4}$$

where $L(\gamma)x$, $Q(\gamma)[x, x]$, and $C(\gamma)[x, x, x]$ are vector valued linear, quadratic, and cubic terms of $f(\gamma, x)$ respectively having symmetric form in each of their entry, and they can each be expanded into

$$\begin{aligned} L(\gamma)x &= L_0 x + \gamma L_1 x + \gamma^2 L_2 x + \cdots, \\ Q(\gamma)[x, x] &= Q_0[x, x] + \gamma Q_1(x, x) + \cdots, \\ C(\gamma)[x, x, x] &= C_0[x, x, x] + \gamma C_1[x, x, x] + \cdots, \end{aligned}$$

where L_0, L_1, and L_2 are $n \times n$ constant matrices.

Let $\tilde{\lambda}$ be the critical eigenvalue of $L(\gamma)$ at the new (bifurcated) equilibria. Then $\tilde{\lambda} = \lambda = 0$ at $\gamma = 0$ due to $L(0) = A(0)$. There exists a series expansion [6]

$$\tilde{\lambda}(\varepsilon) = \sum_{i=0}^{\infty} \tilde{\lambda}_i \varepsilon^i = \tilde{\lambda}_1 \varepsilon + \tilde{\lambda}_2 \varepsilon^2 + \cdots.$$

The computation of the first two coefficients of $\tilde{\lambda}$ can proceed as follows [2]:

- Step 1: Calculate $\lambda'(0) = \ell L_1 r$ where λ is a function of γ.
- Step 2: Set $x_{e1} = r$, and calculate $\gamma_1 = -\ell Q_0[r, r]/\lambda'(0)$.
- Step 3: Compute $x_{e2} = -(\ell^T \ell + L_0^T L_0)^{-1} L_0^T (\gamma_1 L_1 r + Q_0[r, r])$, and
$$\gamma_2 = -\frac{1}{\lambda'(0)} (\gamma_1 \ell L_1 x_{e2} + \gamma_1^2 \ell L_2 r + 2\ell Q_0[r, x_{e2}] + \gamma_1 \ell Q_1[r, r] + \ell C_0[r, r, r]).$$
- Step 4: Set $\tilde{\lambda}_1 = -\gamma_1 \lambda'(0)$ and $\tilde{\lambda}_2 = -2\gamma_2 \lambda'(0)$.

Local stability of the bifurcated solution is given by the following theorem [2].

Theorem 2.1 *Suppose $\gamma_1 \neq 0$. Then the branch of the bifurcated equilibrium solution is locally stable for γ sufficiently close to 0 if $\ell Q_0[r, r]\varepsilon < 0$ and is unstable if $\ell Q_0[r, r]\varepsilon > 0$. For the case $\gamma_1 = 0$, then the bifurcated solution is locally stable for γ sufficiently close to 0 if $\tilde{\lambda}_2 < 0$ and is unstable if $\tilde{\lambda}_2 > 0$, where*

$$\tilde{\lambda}_2 = 2\ell \left(2Q_0[r, x_{e2}] + C_0[r, r, r]\right), \quad x_{e2} = -(\ell^T \ell + L_0^T L_0)^{-1} L_0^T Q_0[r, r].$$

It should be clear that local bifurcation for the case $\gamma_1 \neq 0$ is transcritical [10]. Thus the branch of the bifurcated solution at $\varepsilon > 0$ has the opposite stability property as the one at $\varepsilon < 0$. On the other hand, local bifurcation for the case $\gamma_1 = 0$, and $\gamma_2 \neq 0$ is pitchfork [10], where both branches of the bifurcated solution have the same stability property.

2.2 Local Stability for Hopf Bifurcation

For Hopf bifurcation, it is assumed that $A(\gamma)$ possesses a pair of complex eigenvalues $\lambda(\gamma), \bar{\lambda}(\gamma)$, depend smoothly on γ, while all other eigenvalues are stable in a neighborhood of $\gamma = 0$. Denote $\lambda(\gamma) = \alpha(\gamma) + j\beta(\gamma)$ with $\alpha(\gamma), \beta(\omega)$ real, and $j = \sqrt{-1}$ imaginary. It is assumed that

$$\alpha(0) = 0, \quad \beta(0) = j\omega_c \neq 0, \quad \alpha'(0) = \frac{d\alpha}{d\gamma}(0) > 0. \tag{5}$$

Thus $\lambda(\gamma)$ is a critical eigenvalue, so is its conjugate. It implies that the zero solution $x_e(\gamma) = 0$ is locally stable for $\gamma < 0$ and becomes unstable for $\gamma > 0$. Furthermore Hopf Bifurcation Theorem asserts the existence of a one-parameter family $\{p_\varepsilon\}$, where $0 < \varepsilon \leq \varepsilon_0$, of nonconstant periodic solutions of (1) emerging from the zero solution at $\gamma = 0$. This is none stationary bifurcation. The positive real number ε is a measure of the amplitude of the periodic solution and ε_0 is sufficiently small. The periodic

solutions $p_\varepsilon(t)$ have period near $2\pi/\omega_c$ and occur for parameter values γ given by a smooth function $\gamma(\varepsilon)$. Exactly one of the characteristic exponents of p_ε is near zero, and is given by

$$\tilde{\lambda}(\varepsilon) = \tilde{\lambda}_2\varepsilon^2 + \tilde{\lambda}_4\varepsilon^4 + \cdots = \sum_{i=1}^{\infty} \tilde{\lambda}_{2i}\varepsilon^{2i}. \tag{6}$$

Local stability of Hopf bifurcation is hinged to the first nonzero coefficient of $\tilde{\lambda}(\varepsilon)$, denoted by $\tilde{\lambda}_{2N}$, $N \geq 1$. Generically $N = 1$.

Let the Taylor series of $f(\gamma, x)$ be of the form in (4) where $L_0 = A(0)$. An algorithm to compute $\tilde{\lambda}_2$ is quoted from [1]. See also [5].

- Step 1: Compute left row eigenvector ℓ and right column eigenvector r of L_0 corresponding to the critical eigenvalue of $\lambda(0) = j\omega_c$. Normalize by setting $\ell r = 1$.
- Step 2: Solve column vectors μ and ν from the equations

$$-L_0\mu = \frac{1}{2}Q_0[r, \bar{r}], \quad (2j\omega_c I - L_0)\nu = \frac{1}{2}Q_0[r, r].$$

- Step 3: The coefficient $\tilde{\lambda}_2$ is given by

$$\tilde{\lambda}_2 = 2\mathrm{Re}\left\{2\ell Q_0[r, \mu] + \ell Q_0[\bar{r}, \nu] + \frac{3}{4}\ell C_0[r, r, \bar{r}]\right\}.$$

Theorem 2.2 *Suppose all eigenvalues of L_0 are stable in a neighborhood of $\gamma = 0$ except the critical pair of complex eigenvalues. Then the Hopf bifurcation is stable, if $\tilde{\lambda} < 0$, and is unstable if $\tilde{\lambda} > 0$.*

3 Output Feedback Stabilization for Stationary Bifurcation

We consider first feedback stabilization for stationary bifurcation. The control system in consideration has the form

$$\dot{x} = f(\gamma, x) + g(x)u, \quad y = cx, \tag{7}$$

where $f(\gamma, x)$ is the same as in the previous section and $g(\cdot)$ is also a mooth function. It is assumed that both control input u and output measurement y are scalar functions of time t. The Taylor series expansion of (7) is given by

$$\dot{x} = L_0x + \gamma L_1x + u\tilde{L}_1x + bu + Q_0[x, x] + \gamma^2 L_2x + \gamma Q_1[x, x] + u\tilde{Q}_1[x, x] + C_0[x, x, x] + \cdots, \tag{8}$$

where $\tilde{L}_1x$ and $\tilde{Q}_1[x, x]$ are linear and quadratic components of $g(x)$. It is assumed that L_0 has only one zero eigenvalue with rest of the eigenvalues stable, and that the bifurcated solution born at $\gamma = 0$ is not locally stable. The assumption on stability of the nonzero eigenvalues of L_0 has no loss of generality. If some of the nonzero eigenvalues of L_0 are unstable, then linear control method, such as pole placement [7], can be employed to stabilize those unstable modes corresponding to nonzero eigenvalues. It is the

unstable mode corresponding to the critical eigenvalue $\lambda(0) = 0$ that renders linear control methods inadequate because of bifurcation.

We seek a local output feedback control law

$$u = K(y) = K_1 y + K_2 y^2 + K_3 y^3 + \cdots, \quad K(0) = 0, \quad y = cx, \tag{9}$$

that stabilizes the bifurcated solution. Abed and Fu studied the same problem in [2] for the case of state feedback where the critical mode of L_0 is controllable. We will consider the case of output feedback where the critical mode of L_0 is uncontrollable. It should be clear that in practice, measurement of all state variables is unrealistic. Moreover some of the state variables are more expensive and more difficult to measure than others. Thus often only partial, or a linear combination of, state variables are measurable. Under this circumstance, the critical mode of the linearized system may, or may not be observable based on output measurements. Hence the problem considered in this paper has more engineering significance than that of [2].

With feedback controller in (9), the closed-loop system has the form

$$\dot{x} = L_0^* x + \gamma L_1^* x + Q_0^*[x,x] + \gamma^2 L_2^* x + \gamma Q_1^*[x,x] + C_0^*[x,x,x] + \cdots \tag{10}$$

where the linear, quadratic, and cubic terms are given by

$$\begin{aligned} L_0^* &= L_0 + bK_1 c, \quad L_1^* = L_1, \quad L_2^* = L_2, \\ Q_0^*[x,x] &= Q_0[x,x] + \tilde{L}_1 x K_1 cx + bK_2(cx)^2, \quad Q_1^*[x,x] = Q_1[x,x], \\ C_0^*[x,x,x] &= C_0[x,x,x] + K_2(cx)^2 \tilde{L}_1 x + bK_3(cx)^3 + \tilde{Q}_1[x,x]K_1 cx. \end{aligned}$$

We will establish stabilizability conditions for bifurcated systems where the bifurcated solution is unstable near $\gamma = 0$ in the next two subsections.

3.1 Unobservable Critical Mode

We consider first when the critical mode of linearized system corresponding to the zero eigenvalue at $\gamma = 0$ is not observable through output measurement $y = cx$. Note that by assumption the eigenvalue $\lambda(0) = 0$ is invariant under feedback control because of both uncontrollability and unobservability of the critical mode. Thus L_0^* also possesses the critical zero eigenvalue as L_0. Denote ℓ^* and r^* the left row and right column eigenvectors for L_0^* corresponding to the critical eigenvalue. Then it is easy to see that $\ell^* = \ell$ and $r^* = r$ due to again the uncontrollability and unobservability of the critical eigenvalue. This can be seen from PBH test [7]. Denote $\tilde{\lambda}^*$ as the critical eigenvalue of the linearized feedback system at the bifurcated solution to be stabilized. It is a function of $\varepsilon = \ell^* x_e = \ell x_e$ of the form

$$\tilde{\lambda}^*(\varepsilon) = \tilde{\lambda}_1^* \varepsilon + \tilde{\lambda}_1^* \varepsilon^2 + \cdots. \tag{11}$$

Clearly local feedback controller in (9) does not change the zero solution. Denote $A^*(\gamma)$ as linearized system matrix for the closed-loop system at the zero solution. Its critical eigenvalue is denoted by $\lambda^*(\gamma)$.

Since $L_1^* = L_1$,

$$\frac{d\lambda^*}{d\gamma}(0) = \lambda'(0) = \frac{d\lambda}{d\gamma}(0) > 0$$

is also invariant under feedback control. It follows that the bifurcated solution $x_e \equiv 0$ of the closed-loop system changes its stability and bifurcates at $\gamma = 0$ as well. The problem is whether or not the bifurcated solution can be stabilized with output feedback control. The next result is negative for the transcritical bifurcation.

Theorem 3.1 *Consider the nonlinear control system in (8) with output feedback control law in (9). Suppose that the critical mode of L_0 is not observable through output measurement $y = cx$. Then for the case $\gamma_1 \neq 0$, i.e., $\ell Q_0[r,r] \neq 0$, there does not exist a feedback control law $u = K(y)$, $y = cx$, that stabilizes the given branch of the bifurcated solution, and $\tilde{\lambda}_1^* = \tilde{\lambda}_1$ is invariant under output feedback control in (9).*

Proof: The uncontrollability and unobservability of the critical mode of the linearized system at $\gamma = 0$ imply that both $\ell b = 0$ and $cr = 0$, by PBH test [7]. Hence applying Theorem 2.1 to the nonlinear system in (10) gives the first coefficient of the eigenvalue for the bifurcated solution:

$$\tilde{\lambda}_1^* = \ell Q_0^*[r,r] = \ell Q_0[r,r] + \ell \tilde{L}_1 r K_1 cr + \ell b K_2 (cr)^2 = \ell Q_0[r,r] = \tilde{\lambda}_1,$$

by again $\ell b = 0$ and $cr = 0$ that imply $Q_0^*[r,r] = Q_0[r,r]$. It follows that the sign of $\tilde{\lambda}_1^*$ is the same as $\tilde{\lambda}_1$ that can not be altered by feedback controller in (9). ∎

Although the stability property of transcritical bifurcation can not be altered by output feedback, the situation for pitchfork bifurcation is quite different. We have the following result next.

Theorem 3.2 *Consider the nonlinear control system in (8) with output feedback control law in (9) under the same hypothesis as in Theorem 3.1. Then for the case $\gamma_1 = 0$, i.e., $\tilde{\lambda}_1^* = \tilde{\lambda}_1 = 0$, there exists a feedback control law $u = K(y)$, $y = cx$, that ensures $\tilde{\lambda}_2^* < 0$, i.e., stabilizes the bifurcated solutions, if and only if there exists $K_1 \neq 0$ such that the nonzero eigenvalues of L_0^* remain on the open left half plane and*

$$\tilde{\lambda}_2 + 4\ell \left(Q_0[r, x_{e2}] - Q_0[r, x_{e2}^*]\right) < 0 \tag{12}$$

where K_1 is the linear gain of the feedback controller, and

$$x_{e2} = -(\ell^T \ell + L_0^T L_0)^{-1} L_0^T Q_0[r,r], \quad x_{e2}^* = -(\ell^T \ell + (L_0^*)^T L_0^*)^{-1} (L_0^*)^T Q_0[r,r], \quad L_0^* = L_0 + bK_1 c. \tag{13}$$

If the above conditions hold, then stabilizing feedback controllers can be chosen as linear ones.

Proof: Applying Theorem 2.1 again gives that

$$\begin{aligned}
\tilde{\lambda}_2^* &= 4\ell Q_0^*[r, x_{e2}^*] + 2\ell C_0^*[r,r,r] = 4\ell Q_0[r, x_{e2}^*] + 2\ell C_0[r,r,r] \\
&\quad + 2\ell b \left(2K_2(cr)(cx_{e2}^*) + K_3(cr)^3\right) + 2\ell \left(\tilde{L}_1 x_{e2}^* K_1 cr + \tilde{L}_1 r K_2 (cr)^2 + \tilde{Q}_1[r,r] K_1 cr\right) \\
&= 4\ell Q_0[r, x_{e2}^*] + 2\ell C_0[r,r,r] = \tilde{\lambda}_2 - 4\ell \left(Q_0[r, x_{e2}] - Q_0[r, x_{e2}^*]\right),
\end{aligned}$$

due to again $\ell b = 0$ and $cr = 0$ where x_{e2} and x_{e2}^* are given as in (13). Hence the existence of a stabilizing controller is equivalent to the existence of $K_1 \neq 0$, such that the inequality in (12) is satisfied and the nonzero eigenvalues of L_0^* remain on the open left half plane. ∎

It is interesting to note that although nonlinear feedback controller in (9) is employed, only the linear term has the effect on stability of the bifurcated solution according to Theorem 3.2. Higher order terms are unnecessary if bifurcation stabilization is the sole interest. Hence for bifurcation stabilization with smooth output feedback control considered in this subsection, nonlinear controllers do not offer any advantage over linear ones.

3.2 Observable Critical Mode

Suppose that the stabilizability condition in Theorem 3.2 does not hold. Then bifurcation stabilization with smooth controllers is not possible. In this case we have to consider the case where the critical mode is observable based on output measurement. Extra sensor or sensors have to be deployed so that $cr \neq 0$ is valid. Clearly the nonlinear differential equation (10) holds with output feedback controller in (9) for the case $cr \neq 0$ where the linear, quadratic, and cubic terms, have the same expressions as earlier.

Consider first transcritical bifurcation. Without loss of generality, the branch of $\varepsilon > 0$ is assumed to be unstable for $\gamma > 0$. This is equivalent to $\tilde{\lambda}_1 > 0$. Our goal is to seek a controller of the form (9) that stabilizes the bifurcated solution for $\varepsilon > 0$. It is noted that by assumption the eigenvalue $\lambda(0) = 0$ is invariant under feedback control. Thus L_0^* also possesses the critical zero eigenvalue as L_0 at $\gamma = 0$. Denote ℓ^* and r^* the left row and right column eigenvectors for L_0^* corresponding to the critical eigenvalue. Then it is easy to see that $\ell^* = \ell$ due to the uncontrollability of the critical eigenvalue. Denote $\tilde{\lambda}^*$ as the critical eigenvalue of L_0^* under feedback. It has the same form of the series expansion as in (11). However $r^* \neq r$ in general due to $cr \neq 0$ by the observability of the critical mode. The next result concerns with the stabilization for transcritical bifurcation.

Theorem 3.3 *Consider the nonlinear control system in (8) with output feedback control law in (9). Suppose that the critical mode of the linearized system corresponding to the zero eigenvalue at $\gamma = 0$ is observable, Then for the case $\gamma_1 \neq 0$, i.e., $\ell Q_0[r, r] \neq 0$, there exists a feedback control law $u = K(y)$, $y = cx$, that stabilizes the given branch of the bifurcated solution, if there exist $K_1 \neq 0$ and $r^* \in \mathbf{R}^n$ such that*

$$(L_0 + bK_1c)r^* = 0, \quad \ell r^* = 1, \quad \ell L_1 r^* > 0, \quad \ell Q_0[r^*, r^*] + (\ell \tilde{L}_1 r^*)(K_1 c r^*) < 0, \tag{14}$$

and the rest of nonzero eigenvalues of L_0^* remain in the open left half plane. If these conditions hold, then the zero solution $x_e \equiv 0$ is stable for $\gamma < 0$ and unstable for $\gamma > 0$.

Proof: Suppose that the conditions in (14) hold. Then $L_0^* r^* = (L_0 + bK_1c)r^* = 0$ for some $K_1 \neq 0$ and nonzero vector $r^* \in \mathbf{R}^n$ that implies that r^* is the right eigenvector corresponding to the critical

eigenvalue for the feedback system at $\gamma = 0$. Denote $A^*(\gamma)$ as the system matrix of linearized feedback system at the zero solution, and $\lambda^*(\gamma)$ as the critical eigenvalue. Then

$$\frac{d\lambda^*}{d\gamma}(0) = \ell L_1 r^* > 0$$

at $\gamma = 0$. Hence the zero solution $x_e \equiv 0$ changes its stability at $\gamma = 0$, and the equilibrium solution of the feedback system also bifurcates at $\gamma = 0$. To determine stability of the bifurcated solution, we have by the proof of Theorem 3.1, or an application of Theorem 2.1,

$$\tilde{\lambda}_1^* = \ell Q_0^*[r^*, r^*] = \ell Q_0[r^*, r^*] + \ell \tilde{L}_1 r^* K_1 c r^* + \ell b K_2 (c r^*)^2 = \ell Q_0[r^*, r^*] + (\ell \tilde{L}_1 r^*)(K_1 c r^*) < 0$$

due to $\ell b = 0$ by uncontrollability of the critical mode and the condition of the theorem. It follows that if the rest of nonzero eigenvalues of L_0^* remain in the open left half plane, the bifurcated system is stable. ∎

A natural question is whether or not the equation $(L_0 + bK_1 c)r^* = 0$ admits a nonzero solution $r^* \in \mathbf{R}^n$. Since $\ell b = 0$ and $\ell L_0 = 0$, the rank of $L_0^* = L_0 + bK_1 c$ is exactly $n - 1$ for any K_1 such that the rest of the nonzero eigenvalues of L_o^* remain in the open left half plane. Hence $(L_0 + bK_1 c)r^* = 0$ does admit a nonzero solution $r^* \in \mathbf{R}^n$. The condition $\ell r^* = 1$ is the normalization condition required in using Theorem 2.1, while the condition $\ell L_1^* r^* = \ell L_1 r^* > 0$ guarantees that the zero solution is stable for $\gamma < 0$ and unstable for $\gamma > 0$, and thus $\gamma = 0$ remains a critical value under feedback control. We again note that stabilizing controllers, if exist, can be taken as linear ones because higher order terms in (9) does not have effect on stability of the bifurcated solution.

For pitchfork bifurcation, i.e., $\tilde{\lambda}_1 = 0$, the situation is again different. We adopt an approach as in [2] by setting the linear term of the controller to zero. In fact, by the proof of Theorems 3.1 and 3.3, the nonzero gain K_1 will result in $\tilde{\lambda}_1^* \neq 0$, thereby changing the pitchfork bifurcation into transcritical bifurcation for which only one branch of the bifurcated solution can be stable. Hence this is not a desirable situation unless for some exceptional situations. It is noted that with $K_1 = 0$, the eigenvectors of L_0^* corresponding to the critical eigenvalue at $\gamma = \gamma^*$ satisfy $\ell^* = \ell$ and $r^* = r$. Hence both row and column eigenvectors of the critical eigenvalue are invariant under feedback control. Since $L_1^* = L_1$, there holds $\ell^* L_1^* r^* = \ell L_1 r > 0$. Thus the zero solution of the feedback system changes its stability at $\gamma = 0$. The stabilizability of the bifurcated solution is given by the following result.

Theorem 3.4 *Consider the nonlinear control system in (8) with output feedback control law in (9) under the same hypothesis as in Theorem 3.3. Then for the case $\gamma_1 = 0$, there exists a feedback control law $u = K(y)$, $y = cx$, that ensures $\tilde{\lambda}_1^* = 0$ and $\tilde{\lambda}_2^* < 0$, i.e., changes the pitchfork bifurcation from subcritical into supercritical, if and only if $\ell \tilde{L}_1 r \neq 0$.*

Proof: Suppose that $\ell \tilde{L}_1 r \neq 0$. Then by the proof of Theorem 3.2, there holds

$$\begin{aligned}
\tilde{\lambda}_2^* &= 4\ell Q_0^*[r, x_{e2}^*] + 2\ell C_0^*[r, r, r] = 4\ell Q_0[r, x_{e2}^*] + 2\ell C_0[r, r, r] \\
&\quad + 2\ell b \left(2K_2(cr)(cx_{e2}^*) + K_3(cr)^3\right) + 2\ell \left(\tilde{L}_1 x_{e2}^* K_1 cr + \tilde{L}_1 r K_2 (cr)^2 + \tilde{Q}_1[r, r] K_1 cr\right).
\end{aligned}$$

Setting $K_1 = 0$ gives that $\tilde{\lambda}_1^* = 0$ and $x_{e2}^* = x_{e2}$. Hence we obtain

$$\tilde{\lambda}_2^* = \tilde{\lambda}_2 + 2\ell \tilde{L}_1 r K_2 (cr)^2.$$

Since $cr \neq 0$ and $\ell \tilde{L}_1 r \neq 0$, there exists $K_2 \neq 0$ such that $\tilde{\lambda}_2^* < 0$. Conversely, $\tilde{\lambda}_1^* = 0$ implies that $K_1 = 0$. Moreover stability of the bifurcated solution implies that

$$\tilde{\lambda}_2^* = \tilde{\lambda}_2 + 2\ell \tilde{L}_1 r K_2 (cr)^2 \leq 0,$$

that in turn implies that $\ell \tilde{L}_1 r \neq 0$ by the hypothesis that $\tilde{\lambda}_2 > 0$. ∎

It is noted that terms of higher order than two do not have effect on stability. Thus the stabilizing controllers can be taken as quadratic ones.

4 Output Feedback Stabilization for Hopf Bifurcation

It is assumed that the linearized system matrix $A(\gamma)$ as in (2) has a pair of complex (critical) eigenvalues $\lambda(\gamma) = \alpha(\gamma) \pm j\beta(\gamma)$ such that $\alpha(0) = 0$ and $\alpha'(0) > 0$, while all other eigenvalues are stable. As explained in the previous section, this assumption has no loss of generality. The problem to be investigated in this section is stabilization of Hopf bifurcation with output feedback control, if the Hopf bifurcation born at $\gamma = 0$ for the nonlinear system in (1) is unstable. We consider first the case when the pair of critical modes corresponding to the pair of complex eigenvalues $\lambda(\gamma)$ are both uncontrollable and unobservable. According to PBH test [7], both left and right eigenvectors corresponding to the pair of critical eigenvalues satisfy

$$\ell b = \bar{\ell} b = cr = c\bar{r} = 0.$$

It follows that when the feedback controller (9) is employed, $L_0^* = L_0 + bK_1 c$ retains the pair of the critical eigenvalues $\pm j\omega_c$ at $\gamma = 0$. Denote ℓ^* and r^* as the left and right eigenvectors of L_0^* corresponding to the pair of critical eigenvalues, respectively. Then there hold $\ell^* = \ell$ and $r^* = r$, and thus Hopf bifurcation is again born at $\gamma = 0$ for which the zero solution becomes unstable as γ acrosses 0 from negative to positive [6]. The next result gives the condition on stabilizability of Hopf bifurcation.

Theorem 4.1 *Consider the nonlinear control system in (8) with output feedback control law in (9). Suppose that $\tilde{\lambda}_2 > 0$ with $\tilde{\lambda}(\varepsilon)$ as in (6), and the critical modes of L_0 is not observable through output measurement $y = cx$. Define μ^* and ν^* by*

$$-L_0^* \mu^* = \frac{1}{2} Q_0[r, \bar{r}], \quad (2j\omega_c I - L_0^*)\, \nu^* = \frac{1}{2} Q_0[r, r],$$

where $L_0^* = L_0 + bK_1 c$. Then there exists a feedback control law $u = K(y)$, $y = cx$, that stabilizes Hopf bifurcation, if and only if there exists linear controller $u = K_1 y$, $y = cx$, that stabilizes Hopf bifurcation. Moreover the stabilizing gain $K_1 \neq 0$, if exists, satisfy the condition that the none critical eigenvalues of $L_0^* = L_0 + bK_1 c$ remain in the open left half plane and

$$\tilde{\lambda}_2 + K_1 \text{Re}\left\{\ell \tilde{L}_1 \left(2rc\mu^* + \bar{r}c\nu^*\right)\right\} \le 0. \tag{15}$$

Proof: Denote $\tilde{\lambda}^*(\varepsilon)$ as the function in (6) for controlled system, and $\tilde{\lambda}_2^*$ as the first coefficient of $\tilde{\lambda}^*(\varepsilon)$. Then direct computation gives

$$Q_0^*[x,x] = Q_0[x,x] + \tilde{L}_1 x K_1 cx + bK_2(cx)^2.$$

It follows that $Q_0^*[r,r] = Q_0[r,r]$ and $Q_0^*[r,\bar{r}] = Q_0[r,\bar{r}]$ by $cr = c\bar{r} = 0$. Using the formulas in previous sections and the conditions $\ell b = cr = 0$ yield formula

$$\tilde{\lambda}_2^* = \tilde{\lambda}_2 + K_1 \text{Re}\left\{\ell \tilde{L}_1 \left(2rc\mu^* + \bar{r}c\nu^*\right)\right\}.$$

Since $\tilde{\lambda}_2 > 0$ and $\tilde{\lambda}_2^*$ involves only the linear term of $K(y)$ in (9), stabilization of Hopf bifurcation with nonlinear controllers in (9) is equivalent to the existence of linear stabilizing controllers in light of Theorem 2.2. If stabilizing controllers exist, then $\tilde{\lambda}_2^* \le 0$ holds since $\tilde{\lambda}_2^* = 0$ is possible for stability of Hopf bifurcation under the condition $\tilde{\lambda}_4^* < 0$. Moreover the rest of eigenvalues of $L_0^* = L_0 + bK_1c$ remain in the open left half plane by stability of the closed-loop system. ∎

The result in Theorem 4.1 is similar to stabilizability of pitchfork bifurcation in Theorem 3.2 where nonlinear controllers do not offer any advantages over linear ones in terms of bifurcation stabilization. Hence if the critical mode, or modes, of the linearized system are uncontrollable and unobservable, linear controllers are adequate for bifurcation stabilization. In the rest of the section, we study the case when stabilizability condition in Theorem 4.1 is not satisfied. Clearly additional sensors have to deployed such that $cr \neq 0$ is valid in order for Hopf bifurcation to be stabilizable. Although linear controllers can be investigated, it is much easier to consider the class of nonlinear controllers in (9) where $K_1 = 0$ as discussed in [1]. With $K_1 = 0$, both critical eigenvalues and left/right eigenvectors are invariant under feedback. Moreover $L_0^* = L_0$. The next result generalizes the result in [1] to output feedback stabilization.

Theorem 4.2 *Consider the nonlinear control system in (8) with output feedback control law in (9). Suppose that $\tilde{\lambda}_2 > 0$ with $\tilde{\lambda}(\varepsilon)$ as in (6), μ, ν as in Subsection 2.2, and the critical modes of L_0 is observable through output measurement $y = cx$. Then there exists a feedback control law $u = K(y)$, $y = cx$, that stabilizes the Hopf bifurcation, if and only if*

$$\text{Re}\left\{\ell \tilde{L}_1 \left(\bar{r}cr + rc\bar{r}\right) cr\right\} \neq 0. \tag{16}$$

If the above condition holds, stabilizing controllers can be taken as quadratic ones.

Proof: With straightforward computation and using $\ell b = 0$, we obtain

$$\tilde{\lambda}_2^* = \tilde{\lambda}_2 + K_2 \text{Re}\left\{\ell \tilde{L}_1 \left(2rc\mu + \bar{r}c\nu\right)\right\}.$$

Note that by $K_1 = 0$, $\mu^* = \mu$ and $\nu^* = \nu$. Because $\tilde{\lambda}_2 > 0$, Hopf bifurcation of uncontrolled system is unstable, Hence stabilization requires that $\tilde{\lambda}_2^* < 0$ to hold that implies the condition in (16). Conversely,

if (16) holds, then there exists K_1 such that $\tilde{\lambda}_2^* < 0$ that ensures stability of Hopf bifurcation for the feedback system. Since only quadratic term of the nonlinear controller is involved in determination of $\tilde{\lambda}_2^*$, the stabilizing controller can be taken as quadratic one. ∎

5 Conclusion

This paper discussed bifurcation stabilization using smooth local output feedback controllers where the critical mode of the linearized system is uncontrollable. Stabilizability conditions were established for both the case where the critical mode is observable, and unobservable through output measurement. The results can be used to synthesize stabilizing controllers, if they exist. Although control systems of the form (7) are studied, the results presented in this paper can be easily generalized to those of the form $\dot{x} = f(\gamma, x) + g(\gamma, x)u$, or $\dot{x} = f(\gamma, x, u)$, with suitable modifications.

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